International Journal of Theoretical Physics, Vol. 3, No. 4 (1970), pp. 267–281

Pulsating Stars in General Relativity†

JEFFREY M. COHEN

The Institute for Advanced Study, Princeton, New Jersey 08540

Received: 26 *January* 1970

Abstract

The equation governing radial pulsations of fully relativistic stars is derived and expressed in terms of quantities which are continuous even across density discontinuities which occur, e.g., in zero-temperature stellar models that undergo electron capture. When expressed in terms of these quantities, the pulsation equation can be integrated through density discontinuities without any special treatment of these points being necessary. Expressions for the adiabatic index and pulsation energy are derived in a simple way.

1. *Introduction*

Most Newtonian calculations on stellar models are carried out in Lagrangian (Landau & Lifshitz, 1959) coordinates, since in these coordinates, fixed in the matter composing the star, nuclear physics, opacity and thermodynamic calculations are the easiest to carry out. For example, thermodynamic calculations for moving media are much more complicated than for stationary media. An example of a Lagrangian coordinate is the mass fraction of a star.

Lagrangian coordinates [known as comoving coordinates in general relativity (see for example, Adler *et al.*, 1965)] have similar advantages in general relativistic calculations. However, these coordinates do not seem to have been applied to the problem of stellar model exhibiting small radial pulsations. The pulsation equations for stars exhibiting smooth changes in composition with radius when unperturbed and instantaneous changes in composition with stellar pulsations has been derived in a number of ways by a number of different authors (Taub, 1962; Chandrasekhar, 1964; Cocke, 1965; Harrison *et al.*, 1965). Chandrasekhar, for example, works in Eulerian coordinates but introduces 'Lagrangian displacements'.

In Section 2 the pulsation equation is derived in a straightforward manner using comoving coordinates. In Section 3 the thermodynamic properties of a pulsating material are considered in order to give a simple expression for the adiabatic index γ. Section 4 gives junction conditions which allow the pulsation equations to be used even for stars with sudden changes in

† Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 70-1866.

composition. These junction conditions must be used, for example, when one treats zero-temperature white dwarf models which undergo electron capture (Cohen *et al.*, 1969). Section 5 gives an expression for the pulsation energy of a radially pulsating star.

2. *Pulsation Equation*

In this section we obtain the general relativistic pulsation equation and equilibrium equations for a spherical star undergoing radial pulsations of small amplitude. These equations are obtained by assuming only Einstein's equation, conservation of baryons, spherical symmetry, and an equation of state.

A. *Einstein's Equations*

The line element in comoving coordinates for spherical symmetry takes the form

$$ds^2 = -e^{\nu}\,dt^2 + e^{\lambda}\,dr^2 + R^2(d^2\theta + \sin^2\theta\,d\phi^2) \tag{2.1}$$

Here λ and R are functions of radius r and time t, and ν is a function only of r. If ν were a function of both r and t, the time dependence could be eliminated by changing the time scale. Therefore ν can be assumed to be a function of r only without loss of generality. In the sequel we use the convention $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

For this line element three of Einstein's equations $G^{\mu\nu} = 8\pi T^{\mu\nu}$ become

$$2R_{10} = \lambda_0 R_1 + \nu_1 R_0 \tag{2.2}$$

from the $G^{10} = 8\pi T^{10} = 0$ equation,

$$8\pi\rho = \exp(-\lambda)\left(-2\frac{R_{11}}{R} - \frac{R_1^2}{R^2} + \lambda_1\frac{R_1}{R}\right) + R^{-2} + \exp(-\nu)\left(\frac{R_0^2}{R^2} + \lambda_0\frac{R_0}{R}\right) \tag{2.3}$$

from the G^{00} equation, and

$$8\pi p = \exp(-\lambda)\left(\frac{R_1^2}{R^2} + \nu_1\frac{R_1}{R}\right) - R^{-2} - \exp(-\nu)\left(\frac{2R_{00}}{R} + \frac{R_0^2}{R^2}\right) \tag{2.4}$$

from the G^{11} equation. Here the stress energy tensor for a perfect fluid (Landau & Lifshitz, 1959)

$$T^{\mu\nu} = (\rho + p)\,U^{\mu}\,U^{\nu} + g^{\mu\nu}p \tag{2.5}$$

takes the simple form in comoving coordinates

$$T_0^{\,0} = -\rho \qquad T_1^{\,1} = T_2^{\,2} = T_3^{\,3} = p \tag{2.6}$$

where ρ is the geometrized density (equal to Gc^{-2} times the density in cgs units), p is the geometrized pressure (equal to Gc^{-4} times the pressure in cgs units), and the 4-velocity $U^{\mu} = \delta_0^{\mu}$, since the fluid particles are fixed

relative to the comoving coordinates. The subscripts 0 and 1 denote differentiation with respect to x^0 and x^1, respectively.

B. *Conservation Laws*

Because of the Bianchi identities (Landau & Lifshitz, 1962), Einstein's equations give rise to the conservation laws

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{2.7}$$

The conservation laws take the form

$$0 = \rho_0 + (\rho + p)\left(\frac{\lambda_0}{2} + \frac{2R_0}{R}\right) \tag{2.8}$$

for $\mu = 0$, and

$$p_1 = \frac{-\nu_1(\rho + p)}{2} \tag{2.9}$$

for $\mu = 1$.

C. *Small Pulsations*

For equilibrium stellar model calculations one can assume $R = r$, and the equations (2.2), (2.3), (2.4), and (2.9) above reduce to the usual expression for a non-pulsating star (Cohen *et al.*, 1969). For small pulsations about the equilibrium model we assume

$$R = r + \xi \tag{2.10}$$

where $\xi \ll r$.

The corresponding change in the other quantities are given by

$$\delta\rho = \rho - \bar{\rho} \tag{2.10a}$$

$$\delta p = p - \bar{p} \tag{2.10b}$$

$$\delta\lambda = \lambda - \bar{\lambda} \tag{2.10c}$$

$$\delta\nu = 0 \tag{2.10d}$$

where the barred quantities represented equilibrium values. $\delta\nu$ vanishes, since ν is not a function of time.

Substituting the perturbed quantities of equation (2.10) into equations (2.2), (2.3), (2.4), (2.8) and (2.9) yields, respectively:

$$2\dot{\xi}_{10} = \delta\lambda_0 + \nu_1\,\dot{\xi}_0 \tag{2.11}$$

which integrates to

$$2\dot{\xi}_1 = \delta\lambda + \nu_1\,\xi, \tag{2.11a}$$

$$8\pi\bar{\rho} = \exp(-\bar{\lambda})\left(\frac{\bar{\lambda}_1}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{2.12}$$

the corresponding equation for $\delta\rho$ is not needed in the derivation of the pulsation equation,

$$8\pi\bar{p} = \exp(-\bar{\lambda})\left(\frac{\nu_1}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{2.13}$$

$$8\pi\delta p = \exp(-\bar{\lambda})\left[r^{-2}\left(2\xi_1 - \frac{2\xi}{r}\right) + \frac{\nu_1}{r}\left(\xi_1 - \frac{\xi}{r}\right) - \delta\lambda\left(\frac{1}{r^2} + \frac{\nu_1}{r}\right)\right] + \frac{2\xi}{r^3} - \exp(-\nu)\frac{2\xi_{00}}{r} \tag{2.13a}$$

$$0 = \delta\rho_0 + (\bar{\rho} + \bar{p})\left(\frac{\delta\lambda_0}{2} + \frac{2\xi_0}{r}\right) \tag{2.14}$$

which integrates to

$$\delta\rho = -(\bar{\rho} + \bar{p})\left(\frac{\delta\lambda}{2} + \frac{2\xi}{r}\right) \tag{2.14a}$$

and finally we obtain

$$\bar{p}_1 = \frac{-(\bar{\rho} + \bar{p})\nu_1}{2} \tag{2.15}$$

$$\delta p_1 = \frac{-(\delta\rho + \delta p)\nu_1}{2} \tag{2.15a}$$

Equilibrium stellar models can be obtained via equations (2.12), (2.13) and (2.15) plus an equation of state.

The pulsation equation is obtained by working with equation (2.15a), an equation which came out of the conservation law. Adding to both sides of equation (2.15a) the quantity $[\nu_1 + (\bar{\lambda}_1/2)]\delta p$ and grouping terms yields

$$\exp\left(-\frac{2\nu + \bar{\lambda}}{2}\right)\left[\exp\left(\frac{2\nu + \bar{\lambda}}{2}\right)\delta p\right]_1 = -\frac{\nu_1}{2}\delta\rho + \frac{\nu_1 + \bar{\lambda}_1}{2}\delta p \tag{2.16}$$

The quantity $\nu_1 + \bar{\lambda}_1$ can be eliminated from equation (2.16) by substituting the result of adding equations (2.12) and (2.13)

$$8\pi(\bar{\rho} + \bar{p}) = \exp(-\bar{\lambda})\frac{\nu_1 + \bar{\lambda}_1}{r} \tag{2.17}$$

into equation (2.16), yielding

$$\exp\left(-\frac{2\nu + \bar{\lambda}}{2}\right)\left[\exp\left(\frac{2\nu + \bar{\lambda}}{2}\right)\delta p\right]_1 = -\frac{\nu_1}{2}\delta\rho + 4\pi(\bar{\rho} + \bar{p})\,e^{\bar{\lambda}}\,r\,\delta p \tag{2.18}$$

which yields the pulsation equation once $\delta\rho$ and δp are eliminated.

Substituting equation (2.11a) into equations (2.13a) and (2.14a) yields, respectively,

$$8\pi\delta p = \exp(-\bar{\lambda})\left(-\frac{2\xi}{r^3} - \frac{\nu_1\xi_1}{r} + \frac{\nu_1{}^2\xi}{r}\right) + \frac{2\xi}{r^3} - \frac{2}{r}\exp(-\nu)\xi_{00} \tag{2.19}$$

and

$$\delta\rho = -(\bar{\rho}+\bar{p})\left(\xi_1 + \frac{\nu_1\xi}{2} + \frac{2\xi}{r}\right) \tag{2.20}$$

When equations (2.19) and (2.20) are substituted into equation (2.18) and terms are cancelled it reduces to

$$\exp\left(-\frac{2\nu+\bar{\lambda}}{2}\right)\left[\exp\left(\frac{2\nu+\bar{\lambda}}{2}\right)\delta p\right]_1 = (\bar{\rho}+\bar{p})\left(\frac{\nu_1\xi}{r} + \frac{{\nu_1}^2\xi}{4} + (e^{\bar{\lambda}}-1)\frac{\xi}{r^2} - \exp(\bar{\lambda}-\nu)\xi_{00}\right) \tag{2.21}$$

The quantity $e^{\bar{\lambda}} - 1$ can be eliminated via equation (2.13) and then ν_1 eliminated by using equation (2.15) which reduces equation (2.21) to

$$\exp\left(-\frac{2\nu+\bar{\lambda}}{2}\right)\left[\exp\left(\frac{2\nu+\bar{\lambda}}{2}\right)\delta p\right]_1 = \left(\frac{{\bar{p}_1}^2}{\bar{\rho}+\bar{p}} + \frac{4\bar{p}_1}{r} + 8\pi(\bar{\rho}+\bar{p})\bar{p}\frac{e^{\bar{\lambda}}}{r}\xi - (\bar{\rho}+\bar{p})\exp(\bar{\lambda}-\nu)\xi_{00}\right. \tag{2.22}$$

To put equation (2.22) into Sturm-Liouville form (Margenau & Murphy, 1961), δp must be eliminated from equation (2.22). To do this let us assume an equation of state $n = n(\rho, p)$, where n is the baryon number density. Differentiation yields

$$\delta n = \delta\rho\,\partial_\rho n|_p + \delta p\,\partial_p n|_\rho \tag{2.23}$$

or

$$\delta p = \frac{1}{\partial_p n|_\rho}[\delta n - \delta\rho\,\partial_\rho n|_p] \tag{2.24}$$

This can be expressed in terms of the displacement ξ if one invokes conservation of baryons.

D. *Baryon Conservation*

If there is no mass loss from the stellar model, the relation

$$(nU^\mu)_{;\mu} = 0 \tag{2.25}$$

gives rise to conservation of baryons (Landau & Lifshitz, 1959). This can also be shown by integrating equation (2.25) over a portion of space-time and using the four-dimensional divergence theorem, as has been done elsewhere for a similar problem (Cohen, 1968), carrying out the covariant differentiation in equation (2.23) yields

$$0 = \frac{n_0}{n} + \frac{\lambda_0}{2} + \frac{2R_0}{R} \tag{2.26}$$

which becomes

$$0 = \frac{\delta n}{\bar{n}} + \frac{\delta\lambda}{2} + \frac{2\xi}{r}, \tag{2.27}$$

if we assume small pulsations, equation (2.10), define

$$\delta n = n - \bar{n}, \tag{2.27a}$$

integrate with respect to time, and set the arbitrary function equal to zero.

Substitution of equation (2.11a) into equation (2.27) yields

$$\frac{\delta n}{n} = -\frac{[r^2 \exp(-\nu/2)\,\xi]_1}{r^2 \exp(-\nu/2)} \tag{2.28}$$

after grouping terms. Similarly, substitution of equation (2.11a) into equation (2.14a) yields

$$\delta\rho = -(\bar{\rho} + \bar{p})\frac{[r^2 \exp(-\nu/2)\,\xi]_1}{r^2 \exp(-\nu/2)} \tag{2.29}$$

Substitution of equation (2.28) and (2.29) into equation (2.24) yields

$$\delta p = -\gamma \bar{p}\frac{[r^2 \exp(-\nu/2)\,\xi]_1}{r^2 \exp(-\nu/2)} \tag{2.30}$$

where the function γ is given by

$$\gamma = \frac{[n - (\bar{\rho} + \bar{p})\,\partial_\rho n|_p]}{\bar{p}\,\partial_p n|_\rho} \tag{2.31}$$

The function γ was chosen so that it would satisfy

$$\delta p = \frac{\gamma p \delta\rho}{\rho + p}$$

Substitution of equation (2.30) into equation (2.22) eliminates δp and yields the *pulsation equation in the Sturm-Liouville form.*

Without loss of generality, one can assume that the pulsation amplitude can be represented as a sum of sinusoids since the pulsation equation is linear in ξ. When boundary conditions are imposed, each normal mode obeys an eigenvalue equation with eigenfunction ξ and eigenvalue $2\pi/$(pulsation period for that oscillation mode). The eigenfunctions ξ are orthogonal since the usual orthogenality conditions apply to all equations of the Sturm-Liouville form.

3. *Adiabatic Index*

In this section we express γ in terms of pressure and density alone. This is most easily done in comoving coordinates where the matter is at rest relative to the coordinates. Consequently, the familiar laws of thermodynamics can be used without alteration, and difficulties associated with the thermodynamics of moving media are circumvented.

In this section we will need the relation†

$$T\,ds = dU + p\,dV \tag{3.1}$$

† See Zemansky (1957).

which relates the entropy change dS to the change of internal energy dU and the work done by the system $p\,dV$. Here T is the temperature, p the pressure and V the volume of the system. For convenience, let us consider N particles occupying a volume V at an arbitrary point in the system. For convenience assume that the various quantities are constant throughout the volume V. Thus we have the relations

$$\rho = \frac{U}{V} \qquad \text{and} \qquad n = \frac{N}{V}$$

which yield

$$\frac{T}{V} dS = -(\rho + p)\frac{dn}{n} + d\rho \tag{3.2}$$

since $d(\rho V) + p\,dV = (\rho + p)\,dV + V\,d\rho$.

Solving for n yields

$$n = (\rho + p)\frac{dn}{d\rho} - \frac{T}{V} n \frac{dS}{d\rho} \tag{3.3}$$

Substituting this into equation (2.31) yields

$$\gamma = \frac{\{(\rho + p)\,[(dn/d\rho) - \partial_\rho n|_p] - (T/V)\,n\,(dS/d\rho)\}}{p\,\partial_p n|_\rho}$$

which reduces to

$$\gamma = \frac{[(\rho + p)\,\partial_p n|_\rho\,(dp/d\rho) - (T/V)\,n(dS/d\rho)]}{p\,\partial_p n|_\rho} \tag{3.4}$$

when the chain rule is used $dn/dp = \partial_\rho n|_p + \partial_p n|_\rho\, dp/d\rho$. For constant entropy, the adiabatic index γ takes the simple form

$$\gamma = \frac{\rho + p}{p} \frac{\partial p}{\partial \rho}\bigg|_s \tag{3.5}$$

This relation gives the adiabatic index γ as a function of pressure and density and requires only an equation of state of the form $p = p(\rho)$. It should be noted that the quantity $\partial_\rho p|_s$ is not necessarily given by the static equation of state used to compute the equilibrium models. The ease with which the results of this section were obtained, of course, is due to the use of a reference frame moving with the matter.

4. *Boundary Conditions*

To obtain an equilibrium model one integrates the equations (2.12) (2.13) and (2.15) subject to boundary conditions. Before stating the boundary conditions it is convenient to simplify equation (2.12) and bring it into a form similar to that in the corresponding Newtonian treatment. Combining terms on the right-hand side of Equation (2.12) yields (Cohen & Cohen, 1969)

$$8\pi r^2 \bar{\rho} = \{r[1 - \exp(-\bar{\lambda})]\}_1 \tag{4.1}$$

and the substitution

$$\exp(-\bar{\lambda}) = 1 - 2m(r)r^{-1} \tag{4.2}$$

reduces equation (4.1) to the familiar

$$m_1 = 4\pi r^2 \bar{\rho} \tag{4.3}$$

Here m is the geometrized mass (Harrison *et al.*, 1965) equal to G/c^2 times the mass in units of grams per cubic centimeter. The mass m is in units of centimeters if the gravitational constant G and the light speed c are in cgs units.

At the center of the star we require: $\bar{\rho}$ = some assumed central density (different choices given different models) and $m = 0$, while at the outer boundary of the star we require that $e^{\nu} = 1 - 2mr^{-1}$; $e^{\bar{\lambda}}$ will automatically be continuous if m is.

To determine the frequencies ω of the normal modes of oscillation of a star undergoing adiabatic radial pulsations, one assumes a sinusoidal time dependence $\xi_{00} = -\omega^2 \xi$ in equation (2.22), and imposes boundary conditions. In solving this problem, it is convenient to define two new functions

$$Y = r^2 \exp\left(\frac{\nu}{2}\right)\xi \qquad \text{and} \qquad \xi = \frac{\xi}{r} \tag{4.4}$$

The former is strictly for computational convenience (usually problems of this type must be solved numerically) while the latter ξ is the relative pulsation amplitude. For a discussion for a numerical method for determining these quantities, see, for example, Cohen *et al.*, 1969.

The boundary condition at the center can be found by assuming a power series expression of y about the center giving $y \sim r^3$ or $y \sim$ const. At the center the relative pulsation amplitude ξ must be finite but its value is arbitrary, the value 1 is a convenient choice. Consequently, the boundary condition for y near the center is

$$y = r^3 \exp\left(\frac{\nu}{2}\right) \tag{4.4a}$$

Similarly, expanding y about the outer boundary of the star gives

$$y_1 = y \exp\left(-\frac{3\nu + \lambda}{2}\right)\frac{4e^{\nu} + \omega^2 r^3 m^{-1} + mr^{-1}}{\gamma r} \tag{4.4b}$$

if $p/\rho \to 0$ there. The arguments used in obtaining these boundary conditions are exactly the same as in the neutonian case (Ledoux & Walraven, 1958) when ξ is also introduced (see also, for example, Meltzer & Thorne, 1966; Bardeen *et al.*, 1966).

A. *Junction Conditions*

White dwarf and neutron star models are often idealized as zero-temperature models. This simplifies the physics but often complicates the construction of stellar models. This is because, for example, there are density

discontinuities at points where electron capture takes place. Such points require careful treatment if physically reasonable results are to be obtained.

In general relativity, one matches surfaces across which density discontinuities take place by matching the first and second fundamental forms (Schild, 1967). In comoving coordinates which we are using here, the surface is described by the simple equation $r = \text{const}$. In most other coordinate systems, the calculations to follow, as well as the results, would be much more complicated.

On the surface $r = \text{const.}$, the first fundamental form takes the form:

$$ds^2 = -e^{\nu}\, dt^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{4.5}$$

If the time and angular coordinates are kept the same on both sides of the constant r surface, continuity of the first fundamental form implies that e^{ν} and R^2 be continuous across the surface. In more mathematical terms this condition takes the form:

$$e^{\nu}|_{-}^{+} = 0 \qquad \text{and} \qquad R^2|_{-}^{+} = 0 \tag{4.6}$$

where $+$ and $-$ denote the limits taken from above and below the surface $r = \text{const.}$, respectively. For small oscillations the condition from equation (2.10) implies that ξ is continuous across the surface, i.e.,

$$\xi|_{-}^{+} = 0 \tag{4.7}$$

since the coordinate r is continuous and $R = r + \xi$.

As stated above, other junction conditions are generated by matching the second fundamental form, which is defined by

$$K_{\alpha\beta} = n_{\alpha;\,\beta} \tag{4.8}$$

Here n_{α} is the normal to the surface and the semicolon denotes covariant differentiation in the β direction. Inspection of equation (3.7) shows that the second fundamental form $K_{\alpha\beta}$ describes the rate at which the normal n turns. More important for our purposes, continuity of the second fundamental form guarantees that the derivative of the metric with respect to a coordinate normal to the surface is continuous in a Gaussian normal coordinate system. This condition guarantees that the geometries fit together smoothly (Schild, 1967; Lichnerowitz, 1955).

Carrying out the differentiation indicated in equation (4.8) yields

$$K_{\alpha\beta} = n_{\alpha;\,\beta} = n_{\alpha,\,\beta} + \gamma_{\alpha}\lambda_{\beta}n_{\lambda} \tag{4.9}$$

where the $\gamma_{\alpha}\lambda_{\beta}$ are the usual Christoffel symbols when $K_{\alpha\beta}$ is expressed relative to a coordinate basis (Landau & Lifshitz, 1962). If the frame of reference (i.e. the basis vectors with respect to which the components of vectors are expressed) is chosen so that the basis vectors are orthogonal unit vectors (as is done in vector analysis) and one of these basis vectors

e_1 is chosen along the unit normal n, the components of the normal are not a function of position. Consequently, the ordinary derivatives of n vanish and the second fundamental form takes the simple form

$$K_{\alpha\beta} = \gamma_{\alpha}{}^{1}{}_{\beta} \tag{4.10}$$

When an orthonormal basis is used, the quantities $\gamma_{\alpha}{}^{\lambda}{}_{\beta}$ are called Ricci rotation coefficients, since they describe the rotation of the frame of reference (Brill & Cohen, 1966b).

A convenient set of orthonormal basis vectors ω^{μ} (often called Cartan frames) is

$$\begin{aligned} \omega^0 &= \exp\left(\frac{\nu}{2}\right) dt \\ \omega^1 &= \exp\left(\frac{\lambda}{2}\right) dr \\ \omega^2 &= R\, d\theta \\ \omega^3 &= R \sin\theta\, d\phi \end{aligned} \tag{4.11}$$

Since the Ricci rotation coefficients describe the rotation of the basis vectors, differentiation of the basis vectors ω^{μ} gives the rotation coefficients from the relation

$$d\omega^{\mu} = -\omega^{\mu}{}_{\nu}\, \omega^{\nu} \tag{4.12}$$

where

$$\omega^{\mu}{}_{\nu} = \gamma^{\mu}{}_{\nu\alpha}\, \omega^{\alpha} \tag{4.13}$$

and $\gamma_{\mu\nu\alpha} + \gamma_{\nu\mu\alpha} = 0$.

In other words, the derivative of any basis vector can be expressed in terms of the basis vectors. This is true, since any vector or tensor can be expressed in terms of the basis vectors (definition of the basis vectors).

Carrying out the differentiation of equation (4.11) as indicated in equation (4.12) yields the differentials

$$\begin{aligned} d\omega^0 &= \left[\exp\left(\frac{\nu}{2}\right)\right]_1 dr\, dt = \left[\exp\left(\frac{\nu}{2}\right)\right]_1 \omega^1\, \omega^0 \exp\left(-\frac{\lambda+\nu}{2}\right) \\ d\omega^1 &= \left[\exp\left(\frac{\lambda}{2}\right)\right]_0 dt\, dr = \left[\exp\left(\frac{\lambda}{2}\right)\right]_0 \omega^0\, \omega^1 \exp\left(-\frac{\lambda+\nu}{2}\right) \end{aligned} \tag{4.14}$$

and similarly

$$d\omega^2 = \frac{R_1}{R} \exp\left(\frac{-\lambda}{2}\right) \omega^1\, \omega^2 + \frac{R_0}{R \exp(\nu/2)}\, \omega^0\, \omega^2$$

$$d\omega^3 = \frac{R_1}{R \exp(\lambda/2)}\, \omega^1\, \omega^3 + \frac{R_0}{R \exp(\nu/2)}\, \omega^0\, \omega^3 + \frac{\cot\theta}{R}\, \omega^2\, \omega^3$$

Comparison with equation (4.13) yields

$$\begin{aligned}
&\omega^1{}_0 = \omega^0{}_1 = \left\{\left[\exp\left(\frac{\nu}{2}\right)\right]_1 \omega^0 + \left[\exp\left(\frac{\lambda}{2}\right)\right]_0 \omega^1\right\} \exp\left(-\frac{\lambda+\nu}{2}\right) \\
&\omega^2{}_1 = R_1 \exp\left(\frac{-\lambda}{2}\right) R^{-1}\,\omega^2, \qquad \omega^3{}_1 = R_1\, R^{-1} \exp\left(\frac{-\lambda}{2}\right) \omega^3 \\
&\omega^2{}_0 = R_0 \exp\left(\frac{-\nu}{2}\right) R^{-1}\,\omega^2, \qquad \omega^3{}_0 = R_0\, R^{-1} \exp\left(\frac{-\nu}{2}\right) \omega^3 \\
&\omega^3{}_2 = \frac{\cot\theta}{R}\,\omega^3
\end{aligned} \tag{4.15}$$

The Ricci rotation coefficients can be obtained from equation (4.15) by inspection if one compares with equation (4.13), giving:

$$\begin{aligned}
&\gamma^0{}_{10} = \exp\left(-\frac{\lambda+\nu}{2}\right)\left[\exp\left(\frac{\nu}{2}\right)\right]_1 \\
&\gamma^0{}_{11} = \exp\left(-\frac{\lambda+\nu}{2}\right)\left[\exp\left(\frac{\lambda}{2}\right)\right]_0 \\
&\gamma^2{}_{12} = \gamma^3{}_{13} = R_1\, R^{-1} \exp\left(\frac{-\lambda}{2}\right) \\
&\gamma^2{}_{02} = \gamma^3{}_{03} = R_0\, R^{-1} \exp\left(\frac{-\nu}{2}\right) \\
&\gamma^3{}_{23} = \cot\theta R^{-1}
\end{aligned} \tag{4.16}$$

Comparison of equations (4.10) and (4.16) gives the non-vanishing components of second fundamental form

$$K_{00} = -\exp\left(-\frac{\lambda+\nu}{2}\right)\left[\exp\left(\frac{\nu}{2}\right)\right]_1 = -\frac{\exp(-\lambda/2)\,\nu_1}{2} \tag{4.17}$$

$$K_{22} = K_{33} = R_1\, R^{-1} \exp\left(\frac{-\lambda}{2}\right) \tag{4.17a}$$

Consequently, the quantities

$$\exp\left(\frac{-\lambda}{2}\right)\nu_1 \qquad \text{and} \qquad R_1\, R^{-1} \exp\left(\frac{-\lambda}{2}\right)$$

are continuous across the density discontinuity. In concise terms the additional junction conditions become

$$\exp\left(\frac{-\lambda}{2}\right)\nu_1\Big|_-^+ = 0 \qquad \text{and} \qquad R_1\, R^{-1} \exp\left(\frac{-\lambda}{2}\right)\Big|_-^+ = 0 \tag{4.18}$$

Substitution of the junction conditions (4.18) and (4.6) into equation (2.4) yields a junction condition in terms of the physical quantity pressure:

$$p|_{-}^{+} = 0 \tag{4.19}$$

Thus, as in neutonian mechanics, the pressure must be continuous in Lagrangian (comoving) coordinates.

From equation (2.10b) we have $p = \bar{p} + \delta p$; consequently, $\bar{p}$ and δp are continuous if equation (4.19) is satisfied in each order. The condition

$$\delta p|_{-}^{+} = 0 \tag{4.20}$$

implies that the right-hand side of equation (2.30) is continuous across the density discontinuity. This, together with the requirement that ξ be continuous, allows the eigenvalue equation (2.22) to be integrated across density discontinuities without any special treatment of these surfaces. For a discussion of numerical methods for treating such problems see, for example, Cohen *et al.* (1969) and the references cited there.

5. *Pulsation Energy*

In general relativity, the pulsation energy of a star can be found by finding the total energy of the pulsating star and subtracting the energy of the same star when it is not pulsating. The pulsation energy has been given by a number of authors [see, for example, Harrison *et al.* (1965)], but for completeness we will derive it here in a new and simpler way using the standard Schwarzschild line element

$$ds^2 = \exp(-\nu)\, dt^2 + e^{\lambda}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \tag{5.1}$$

Here the fluid moves relative to the coordinates which are fixed in space. The quantities ν and λ are functions of r only.

In these coordinates, the total energy (Schwarzschild mass) of the star is given by (Landau & Lifshitz, 1962)

$$m = -\int T_0{}^0 r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{5.2}$$

The calculations simplify considerably if one notes that since the pulsation energy is constant during the adiabatic pulsations, it can be computed at any instant. In particular, the instant when the star passes through its equilibrium configuration is especially convenient, since all the potential energy terms vanish and only the kinetic energy terms contribute. At that instant, the stress-energy tensor (2.5) takes the form

$$-T_0{}^0 = \rho + (\bar{\rho} + \bar{p}) \exp(\bar{\lambda} - \bar{\nu})\, \dot{\xi}^2 \tag{5.3}$$

to second order in $\dot{\xi}$. Substituting the expression (5.3) for the stress-energy tensor into the expression for the total energy (5.2) yields

$$m = 4\pi \int (\rho + (\bar{\rho} + \bar{p}) \exp(\bar{\lambda} - \bar{\nu})\, \dot{\xi}^2)\, r^2\, dr \tag{5.4}$$

Here the barred quantities $\bar{\rho}$, $\bar{p}$, etc., are unperturbed quantities, while the total mass m and the density ρ have contributions from the kinetic energy of pulsation.

The mass and density can be expressed as an unperturbed value plus a perturbation

$$m = \bar{m} + m_2 \tag{5.5}$$
$$\rho = \bar{\rho} + \rho_2$$

Terms of higher order than second have been dropped.

The Schwarzschild mass of the unperturbed star $\bar{m}$ and the pulsational energy m_2 are obtained by substitution of equation (5.5) into (5.4), giving, after collecting terms of the same order,

$$\bar{m} = 4\pi \int \bar{\rho} \bar{r}^2 \, d\bar{r} \tag{5.6}$$
$$m_2 = 4\pi \int \bar{r}^2 \, d\bar{r} [\rho_2 + (\bar{\rho} + \bar{p}) \exp(\bar{\lambda} - \bar{\nu}) \dot{\xi}^2]$$

Thus, the problem has been reduced to finding ρ_2 which can be expressed in terms of the baryon number density n via equation (3.2)

$$\rho_2 = \frac{(\bar{\rho} + \bar{p}) n_2}{\bar{n}} \tag{5.7}$$

since the entropy is constant for adiabatic pulsations.

To find the baryon number density n_l in the frame of the metric (5.1), one can use the relation

$$n \, dV = n_l \, dV_l \tag{5.8}$$

between the number density n in the rest frame of the matter and that in the frame of the metric (5.1). Since the 4-volume element is invariant, with respect to coordinate transformations, the 3-volume elements are related by

$$\sqrt{(|g|)} \, dV_l \, dt = \exp\left(\frac{\nu}{2}\right) dV_l \, dt = dV \, d\tau \tag{5.9}$$

the number densities are related by

$$n = n_l \exp\left(\frac{-\nu}{2}\right) \frac{d\tau}{dt} \tag{5.10}$$

where

$$d\tau = \exp\left(\frac{\nu}{2}\right) dt [1 - \exp(\lambda - \nu) \dot{\xi}^2]^{1/2} \tag{5.11}$$

The baryon number density n_l is defined as

$$n_l = \frac{dA}{dV_l} = \frac{dA}{4\pi r^2 \, dr \exp(\lambda/2)} \tag{5.12}$$

where A is the baryon number (an invariant). Here the volume element dV_l is not equal to the volume element $d\bar{V}$ of the unperturbed star. This

is because the pulsational kinetic energy contributes to the mass. Consequently, we obtain

$$\exp(-\lambda) = 1 - \frac{2m}{r} = 1 - \frac{2\bar{m}}{r} - \frac{2m_2}{r}$$

or

$$\exp\left(\frac{-\lambda}{2}\right) = \exp\left(\frac{-\bar{\lambda}}{2}\right)\left(1 - \frac{2m_2}{r}e^{\lambda}\right)^{1/2} \tag{5.13}$$

Thus, the baryon number density n_l is related to the number density

$$\bar{n} = \frac{dA}{4\pi\bar{r}^2\, d\bar{r} \exp(\bar{\lambda}/2)}$$

of the unperturbed star by

$$n_l = n_0\left(1 - \frac{2m_2}{r}e^{\bar{\lambda}}\right)^{1/2} \tag{5.14}$$

Substitution of equations (5.11) and (5.14) into (5.12) yields

$$n = n_0\left(1 - \frac{2m_2}{r}e^{\bar{\lambda}}\right)^{1/2}[1 - \exp(\lambda - \nu)\dot{\xi}^2]^{1/2} \tag{5.15}$$

Expansion of equation (5.15) yields

$$-(n_2/\bar{n}) = n_2\bar{r}^{-1}\,\mathrm{e}^{\bar{\lambda}} + \tfrac{1}{2}\exp(\bar{\lambda} - \bar{\nu})\dot{\xi}^2 \tag{5.16}$$

The pulsational energy is obtained by substitution of equation (5.16) into (5.7), and the resulting expression into (5.6), yielding the integral equation

$$m_2 = 4\pi\int_0^{r_0}\bar{r}^2\, d\bar{r}(\bar{\rho} + \bar{p})\left[\exp(\bar{\lambda} - \bar{\nu})\frac{\dot{\xi}^2}{2} - m_2\bar{r}^{-1}\,\mathrm{e}^{\bar{\lambda}}\right] \tag{5.17}$$

This equation can be solved by converting it into a differential equation, which can be accomplished via differentiation with respect to r_0, yielding:

$$(m_2)_{r_0} + 4\pi r_0(\bar{\rho} + \bar{p})\,\mathrm{e}^{\bar{\lambda}}\, m_2 = 4\pi r_0^2(\bar{\rho} + \bar{p})\frac{\exp(\bar{\lambda} - \bar{\nu})\dot{\xi}^2}{2} \tag{5.18}$$

The left-hand side can be expressed as a derivative by using equation (2.17) once again and integrating

$$\int_0^{\infty}\left[\exp\left(\frac{\bar{\lambda} + \bar{\nu}}{2}\right)m_2\right]_r dr = 2\pi\int_0^{\infty} r^2\, dr(\bar{\rho} + \bar{p})\exp\left(\frac{3\bar{\lambda} - \bar{\nu}}{2}\right)\dot{\xi}^2 \tag{5.19}$$

yielding the expression for the pulsation energy.

$$m_2 = 2\pi\int r^2\, dr(\bar{\rho} + \bar{p})\exp\left(\frac{3\bar{\lambda} - \bar{\nu}}{2}\right)\dot{\xi}^2 \tag{5.20}$$

In the newtonian limit, this expression reduces to that of Ledoux & Walraven (1958).

6. *Discussion*

In Section 2 the radial pulsation equation (2.22) was derived using comoving (Lagrangian) coordinates. By the use of these coordinates, it was possible to obtain the adiabatic index (3.5) in a simple manner using standard relations from thermodynamics. No account of mass motion had to be taken, since there is none relative to the comoving coordinates. In Section 4 it was shown that the quantities ξ, δp [equation (2.30)] e^{ν} and R^2 are continuous across density discontinuities. Consequently, if the pulsation equation is expressed in terms of the dependent variables δp and $r^2 \exp(-\nu/2)\xi$ (which are continuous across density discontinuities), it can be integrated across density discontinuities without any special treatment of these points being necessary. For example, with zero-temperature white dwarf and neutron star models, one does not have to integrate out from the center and in from the surface and match across the discontinuity. One can merely integrate out from the center and stop at the surface. This is especially convenient with stellar models which exhibit electron capture at a number of different thresholds, and consequently show density discontinuities each time A/Z changes. This method has been applied to white dwarf models (Cohen *et al.*, 1969). A simple derivation of the pulsation energy is given in Section 5.

References

Adler, R., Bazin, M. and Schiffer, M. (1965). *Introduction to General Relativity*. McGraw-Hill, New York.

Bardeen, J., Thorne, K. S. and Meltze, D. (1966). *Astrophysical Journal*, **145**, 505.

Brill, D. R. and Cohen, J. M. (1966). *Journal of Mathematics and Physics*, 7, 238.

Chandrasekhar, S. (1964). *Astrophysical Journal*, **140**, 417.

Cocke, W. J. (1965). *Annales de l'Institut Henri Poincaré*, **4**, 238.

Cohen, J. M. and Cohen, M. D. (1969). *Nuovo cimento*, **60B**, 241.

Cohen, J. M., Lapidus, A. and Cameron, A. G. W. (1969). *Astrophys. & Space Sci.*, **5**, 113.

Harrison, B. K., Thorne, K. S., Wakano, M. and Wheeler, J. A. (1965). *Gravitation Theory and Gravitational Collapse*. University of Chicago Press.

Landau, L. D. and Lifshitz, E. (1959). *Statistical Physics*. Addison-Wesley Publishing Company, Reading, Mass.

Landau, L. D. and Lifshitz, E. M. (1962). *The Classical Theory of Fields*. Addison-Wesley Publishing Company, Reading, Mass.

Ledoux, P. and Walraven, Th. (1958). In: *Handbuch der Physik*, pp. 51, 458, ed. Flugge, S. Springer-Verlag, Berlin.

Lichnerowicz, A. (1955). *Theories relativistes de la gravitation et de l'electromagnetisme*. Masson, Paris.

Margenau, H. and Murphy, G. M. (1961). *The Mathematics of Physics and Chemistry*. Van Nostrand, New York.

Meltzer, D. W. and Thorne, K. S. (1966). *Astrophysical Journal*, **145**, No. 2, 514.

Schild, A. (1967). *Lectures in Applied Mathematics, Relativity Theory and Astrophysics*, Vol. 8. American Math. Soc., Providence.

Taub, A. H. (1962). *Colloques internationaux du Centre National de la recherche scientifique*, **91**, 173.

Zemansky, M. W. (1957). *Heat and Thermodynamics*. McGraw-Hill, New York.